On Projectively Related Einstein Metrics

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Abstract

In this paper we study pointwise projectively related Einstein metrics (having the same geodesics as point sets). We show that pointwise projectively related Einstein metrics satisfy a simple equation along geodesics. In particular, we show that if two pointwise projectively related Einstein metrics are complete with negative Einstein constants , then one is a multiple of another.

1 Introduction

Two regular metrics on a manifold are said to be *pointwise projectively related* if they have the same geodesics as point sets. Two regular metric spaces are said to be *projectively related* if there is a diffeomorphism between them such that the pull-back metric is pointwise projective to another one. Regular metrics under our consideration are Finsler metrics neither necessarily Riemannian nor reversible. A Riemann metric has two features: length and angle, while a Finsler metric has only one feature: length. In projective metric geometry, the second feature of Riemann metrics is generally not used. Thus, we do not restrict our attention to Riemannian metrics.

There are many Finsler metrics on a strongly convex subset $\Omega \subset \mathrm{R}^n$ which are pointwise projective to the standard Euclidean metric (such Finsler metrics are simply called the *projective Finsler metrics*). The problem of characterizing and studying projective Finsler metrics is known as *Hilbert's fourth problem*. In dimension 2, Darboux gave a general formula for projective Finsler metrics [Da][M2]. R. Alexander's paper [Ale] treats the planar case of Hilbert's fourth problem by a direct geometric argument which is entirely free from the notion of differentiability. In [Al1], J.C. Álvarez presents two constructions of projective metrics on R^n. In [AGS], Álvarez-Gelfand-Smirnov presents more comprehensive constructions of projective metrics on an open convex subset $\Omega \subset \mathrm{R}^n$. See also [Al2]. Although there are many projective metrics, all of them are of scalar curvature. If a projective Finsler metric is an Einstein metric, then it must be of constant curvature. See Section 2 below for more details.

On an arbitrary strongly convex domain $\Omega \subset \mathrm{R}^n$, there is a pair of Funk metrics $F_\pm$. The symmetrization $F_H := \frac{1}{2}(F_- + F_+)$ is called the *Hilbert metric* on Ω. It is known that $F_\pm$ are positively/negatively complete with constant curvature $-1/4$, while F_H is complete with constant curvature -1. All of them are pointwise projective to the Euclidean metric $F_E(y) := |y|$. See [Ok] for a beautiful proof on this fact. The pair of Funk metrics on the unit ball $\mathrm{B}^n \subset \mathrm{R}^n$ are given by

$$F_\pm(y) := \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)} \pm \langle x, y\rangle}{1 - |x|^2}, \tag{1}$$

where $|\cdot|$ and $\langle\ ,\ \rangle$ denotes the standard Euclidean norm and inner product. Note that the Hilbert metric $F_H = \frac{1}{2}(F_- + F_+)$ on B^n is just the Klein metric F_K given by

$$F_K(y) := \frac{\sqrt{|y|^2 - \Big(|x|^2|y|^2 - \langle x, y\rangle^2\Big)}}{1 - |x|^2}. \tag{2}$$

A natural problem is to determine all projective metrics of constant curvature on a given open subset in R^n. More general, given a Finsler metric on a manifold M, we would like to determine all Finsler metrics which are pointwise projective to the given one. There are several papers on this problem, especially in Riemann geometry. See [Mi] for a survey.

In this paper, we study the following problem: given an Einstein metric, describe all Einstein metrics which are pointwise projective to the given one. Below are our main theorems.

Theorem 1.1 *Let F and $\tilde{F}$ be Einstein metrics on a closed n-manifold M with*

$$\mathbf{Ric} = (n-1)\lambda, \qquad \widetilde{\mathbf{Ric}} = (n-1)\tilde{\lambda},$$

where $\lambda, \tilde{\lambda} \in \{-1, 0, 1\}$. Suppose that $\tilde{F}$ is pointwise projectively related to F. Then λ and $\tilde{\lambda}$ have the same sign. More details are given below.

(i) If $\lambda = 1 = \tilde{\lambda}$, then along any unit speed geodesic $c(t)$ of F

$$\tilde{F}(\dot{c}(t)) = \frac{2}{\Big(a^2 - 1/a^2 - b^2\Big)\cos(2t) + 2ab\sin(2t) + \Big(a^2 + 1/a^2 + b^2\Big)}, \tag{3}$$

where $a > 0$ and $-\infty < b < \infty$ are constants. Thus, for any unit speed geodesic segment c of F with length of π, it is also a geodesic segment of $\tilde{F}$ (as a point set) with length of π.

(ii) If $\lambda = 0 = \tilde{\lambda}$, then along any geodesic $c(t)$ of F or $\tilde{F}$,

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant. \tag{4}$$

(iii) If $\lambda = -1 = \tilde{\lambda}$, then

$$\tilde{F} = F.$$

Theorem 1.1 is stated for Finslerian Einstein metrics. Since Riemann metrics are special Finsler metrics, this is of course true for Riemannian Einstein metrics.

By Theorem 1.1(i), we know that for any Einstein metric F with $\mathbf{Ric} = n-1$ on S^n, if the geodesics of F are great circles on S^n, then along any unit speed great circle $c(t)$ on S^n, $F(\dot{c}(t))$ must be in the form (3) and the F-length of c is still equal to 2π.

According to [Br1][Br2], there are lots of non-reversible Finsler metrics F of constant curvature $K = 1$ on S^2 whose geodesics are great circles as point sets with F-length of 2π. The later is also guaranteed by Theorem 1.1(i). In Example 7.2 below, we shall extend Bryant's metrics to higher dimensional spheres.

Theorem 1.1 (ii) is an almost projective rigidity result for Ricci-flat metrics. In general, Ricci-flat metrics are not projectively isolated. For example, locally Minkowski metrics on a torus T^n are pointwise projective to the standard flat Riemann metric on T^n. In fact, these are the only flat metrics on T^n.

Theorem 1.1 (iii) is a projective rigidity result for negative Einstein metrics. Any negative Einstein metric on a closed manifold is projectively isolated.

Theorem 1.2 *Let F and $\tilde{F}$ be Ricci-flat metrics on a non-compact n-manifold M. Suppose that F and $\tilde{F}$ are pointwise projectively related. Then along any unit speed geodesic $c(t)$ of F,*

$$\tilde{F}(\dot{c}(t)) = \frac{1}{(a+bt)^2}, \tag{5}$$

where $a > 0$ and $-\infty < b < \infty$ are constants. Then F is complete if and only if $\tilde{F}$ is complete. In this case, along any geodesic $c(t)$ of F or $\tilde{F}$,

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = constant.$$

Any Minkowski metric on R^n is pointwise projective to the standard Euclidean metric on R^n.

We actually prove that if c is a unit speed geodesic of F, along which $F/\tilde{F} \neq$ *constant*, then c can not be defined on $(-\infty, \infty)$. If c is defined on $[0, \infty)$, then it has finite $\tilde{F}$-length. If c is defined on $(-\infty, 0]$, then it has finite $\tilde{F}$-length. This suggests that there might be a positively (Ricci-)flat metric and a negatively complete (Ricci-)flat metric which are pointwise projective to each other (they might be also pointwise projective to the standard Euclidean metric on an open subset $\Omega \subset \mathrm{R}^n$). Such examples have not been found yet.

Theorem 1.3 *Let F and $\tilde{F}$ be Einstein metrics on a non-compact n-manifold with*

$$\mathbf{Ric} = -(n-1), \qquad \widetilde{\mathbf{Ric}} = -(n-1).$$

Suppose that $\tilde{F}$ is pointwise projectively related to F, then along any geodesic $c(t)$ in (M, F),

$$\tilde{F}(\dot{c}(t)) = \frac{2}{\Big(-1/a^2 + a^2 + b^2\Big)\cosh(2t) + 2ab\sinh(2t) - \Big(-1/a^2 - a^2 + b^2\Big)}, \tag{6}$$

where $a > 0$ and $-\infty < b < \infty$ are constants.

(i) If both F and $\tilde{F}$ are complete. Then

$$F = \tilde{F}.$$

(ii) If F is complete, then for any geodesic c of F, the $\tilde{F}$-length of c is finite unless

$$\tilde{F}(\dot{c}(t)) = \frac{1}{e^{\pm 2t}(a^2-1)+1}, \qquad (a \geq 1). \tag{7}$$

According to Theorem 1.3, if a Finsler metric F is a complete projective metric on an open subset $\Omega \subset \mathrm{R}^n$ with constant curvature -1, then it must be the Hilbert metric F_H on Ω. This fact was proved earlier in [Ber][Fk]. Theorem 1.3 also gives us some information on incomplete projective metrics of constant curvature -1.

One can verify that for $F := F_H$ and $\tilde{F} := F_\pm$, along any unit speed geodesic $c(t)$ of F, $\tilde{F}(\dot{c}(t))$ is in the form (7). Further discussions on examples are given in §7 below.

2 Preliminaries

Let M be an n-dimensional manifold. A Finsler metric under our consideration is a function $F : TM \to [0, \infty)$ with the following properties.

(i) F is C^∞ on $TM - \{0\}$,

(ii) For any $x \in M$, $F_x := F|_{T_xM}$ is a Minkowski functional, namely,

(iia) F_x is positively homogeneous of degree one

$$F_x(\lambda y) = \lambda F_x(y), \qquad \lambda > 0,\ y \in T_xM$$

(iib) for any $y \in T_xM - \{0\}$, the Hessian $g_{ij}(y)$ of $\frac{1}{2}F^2$ at y is positive definite, where

$$g_{ij}(y) := \frac{1}{2}\frac{\partial^2[F^2]}{\partial y^i \partial y^j}(y).$$

For any $y \in T_xM - \{0\}$, the Hessian $(g_{ij}(y))$ induces an inner product g_y in T_xM by

$$g_y(u, v) := g_{ij}(y)u^i v^j.$$

If g is a Riemann metric, then

$$F(y) := \sqrt{g(y,y)}$$

is a Finsler metric.

Every Finsler metric induces a spray $\mathbf{G}$ on M

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(y)\frac{\partial}{\partial y^i}, \tag{8}$$

where

$$G^i(y) := \frac{1}{4} g^{il}\Big\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \Big\}. \tag{9}$$

$\mathbf{G}$ is a globally defined vector field on TM. The projection of a flow line of $\mathbf{G}$ is called a *geodesic* in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(x^i(t))$ satisfy

$$\ddot{c}^i + 2G^i(\dot{c}) = 0. \tag{10}$$

F is said to be *positively complete* (resp. *negatively complete*), if any geodesic on an open interval (a,b) can be extended to a geodesic on (a,∞) (resp. $(-\infty,b)$). F is said to be *complete* if it is positively and negatively complete. There are Finsler metrics which are positively complete, but not complete. See Example 7.4 below.

The notion of Riemann curvature for Riemann metrics can be extended to Finsler metrics/sprays. For a vector $y \in T_xM - \{0\}$, the Riemann curvature $\mathbf{R}_y : T_xM \to T_xM$ is defined by

$$\mathbf{R}_y(u) = R^i_{\ k}(y) u^k \ \frac{\partial}{\partial x^i},$$

where

$$R^i_{\ k}(y) := 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{11}$$

Take an arbitrary plane $P \subset T_xM$ (flag) and a non-zero vector $y \in P$ (flag pole), the *flag curvature* $K(P,y)$ is defined by

$$K(P,y) := \frac{g_y(\mathbf{R}_y(v), v)}{g_y(y,y)g_y(v,v) - g_y(v,y)g_y(v,y)}. \tag{12}$$

F is said to be *scalar curvature* if for any nonzero vector $y \in T_xM$ and any flag $P \subset T_xM$, $x \in M$, with $y \in P$, $K(P,y) = \lambda(y)$ is independent of P, or equivalently,

$$\mathbf{R}_y = \lambda(y) F^2(y) \ \Big\{ I - g_y(y,\cdot) y \Big\}, \qquad y \in T_xM,\ x \in M,$$

where $I : T_xM \to T_xM$ denotes the identity map and $g_y(y, \cdot) = -\frac{1}{2}[F^2]_{y^i}dx^i$. It is said to be *of constant curvature* λ if the above identity holds for the constant λ.

The trace of the Riemann curvature $\mathbf{R}_y$ is a scalar function $\mathbf{Ric}$ on TM

$$\mathbf{Ric}(y) := \text{trace of } \mathbf{R}_y. \tag{13}$$

$\mathbf{Ric}$ is called the *Ricci curvature*. Let $R = \frac{1}{n-1}\mathbf{Ric}$.

A Finsler metric is called an *Einstein metric with Einstein constant* λ if

$$\mathbf{Ric}(y) = (n-1)\,\lambda\,F^2(y). \tag{14}$$

(14) is simply denoted by $\mathbf{Ric} = (n-1)\lambda$ if no confusion is caused.

We now consider pointwise projectively related Finsler metrics — those having the same geodesics as set points. Given two Finsler metrics F and $\tilde{F}$ on an n-dimensional manifold M, let $\mathbf{G}$ and $\tilde{\mathbf{G}}$ be the sprays induced by F and $\tilde{F}$, respectively. It is easy to verify that

$$\tilde{G}^i = G^i + \frac{\tilde{F}_{;k}y^k}{2\tilde{F}}\,y^i + \frac{\tilde{F}}{2}\tilde{g}^{il}\Big\{\frac{\partial \tilde{F}_{;k}}{\partial y^l}y^k - \tilde{F}_{;l}\Big\}, \tag{15}$$

where $\tilde{F}_{;k}$ denotes the covariant derivatives of $\tilde{F}$ on (M, F).

$$\tilde{F}_{;k} := \frac{\partial \tilde{F}}{\partial x^k} - \frac{\partial G^l}{\partial y^k}\frac{\partial \tilde{F}}{\partial y^l}. \tag{16}$$

The identity (15) was first established by A. Rapcsák [Rap]. By (15), Rapcsák proved the following important lemma

Lemma 2.1 (Rapcsák) *Let (M, F) be a Finsler space. A Finsler metric $\tilde{F}$ is pointwise projective to F if and only if*

$$\frac{\partial \tilde{F}_{;k}}{\partial y^l}y^k - \tilde{F}_{;l} = 0. \tag{17}$$

In this case,

$$\tilde{G}^i = G^i + Py^i \tag{18}$$

with

$$P = \frac{\tilde{F}_{;k}y^k}{2\tilde{F}}. \tag{19}$$

By Rapcsák's lemma, we conclude that a Finsler metric $\tilde{F}$ on an open subset $\Omega \subset \mathrm{R}^n$ is a projective metric if and only if

$$\frac{\partial \tilde{F}}{\partial x^k \partial y^l}y^k - \frac{\partial \tilde{F}}{\partial x^l} = 0. \tag{20}$$

One can verify that the Klein metric F_K and the Funk metrics $F_\pm$ in Section 1.1 satisfy (20). Hence they are projective metrics.

Let F and $\tilde{F}$ be Finsler metrics on an n-dimensional manifold M. Assume that $\tilde{F}$ is pointwise projective to F, i.e., it satisfies (17). Plugging (18) into (11) yields

$$\begin{aligned}\tilde{\mathbf{R}}_y(u) &= \mathbf{R}_y(u) + \Xi(y)\, u + \tau_y(u)\, y, & (21)\\ \widetilde{\mathbf{Ric}}(y) &= \mathbf{Ric}(y) + (n-1)\Xi(y), & (22)\end{aligned}$$

where

$$\begin{aligned}\Xi(y) :&= P^2 - P_{;k}y^k, & (23)\\ \tau_y(u) :&= 3\Big(P_{;k} - \frac{1}{2}\frac{\partial [P^2]}{\partial y^k}\Big)u^k + \frac{\partial \Xi}{\partial y^k}u^k, & (24)\end{aligned}$$

where $P_{;k}$ denote the covariant derivatives of P on (M, F) as defined in (16) for $\tilde{F}$. Using (19), one can express $\Xi(y)$ and τ_y in terms of $\tilde{F}$ and its covariant derivatives on (M, F). These formulas are given in [M2][MW].

We immediately obtain the following

Proposition 2.1 *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M.*

(i) Assume that $\mathbf{Ric} = (n-1)\lambda$. Then $\widetilde{\mathbf{Ric}} = (n-1)\tilde{\lambda}$ if and only if

$$\Xi = \tilde{\lambda}\tilde{F}^2 - \lambda F^2. \qquad (25)$$

(ii) Assume that $\mathbf{R} = \lambda$. Then $\tilde{\mathbf{R}} = \tilde{\lambda}$ if and only if (25) holds.

Proposition 2.1(ii) is proved in [MW].

There is a simple sufficient condition for $\tilde{F}$ being of negative constant curvature and pointwise projective to F.

Proposition 2.2 *Let (M, F) be a Finsler space of dimension n and $\tilde{F}$ another Finsler metric on M. Suppose that*

$$\tilde{F}_{;k} = \mu \frac{\partial [\tilde{F}^2]}{\partial y^k}, \qquad (26)$$

where μ is a constant. Then $\tilde{F}$ is pointwise projective to F and

$$\begin{aligned}\tilde{\mathbf{R}}_y(u) &= \mathbf{R}_y(u) - \mu^2\Big[\tilde{F}^2(y)u - \tilde{g}_y(y,u)y\Big], & (27)\\ \widetilde{\mathbf{Ric}}(y) &= \mathbf{Ric}(y) - (n-1)\mu^2\tilde{F}^2(y), & (28)\end{aligned}$$

where $\tilde{g}_y(y,u) := \frac{1}{2}\frac{\partial[\tilde{F}^2]}{\partial y^k}(y)u^k$. Hence,

(i) if F is Ricci-flat($\mathbf{Ric} = 0$), then $\tilde{F}$ is an Einstein metric with $\widetilde{\mathbf{Ric}} = -(n-1)\mu^2$;

(ii) if F is R-flat ($\mathbf{R} = 0$), then $\tilde{F}$ is of constant curvature with $\tilde{\mathbf{R}} = -\mu^2$.

Proof: Differentiating (26) with y^l, we obtain

$$\frac{\partial \tilde{F}_{;k}}{\partial y^l} = \mu \frac{\partial^2[\tilde{F}^2]}{\partial y^k \partial y^l}. \tag{29}$$

Contracting (29) with y^k and using (26) again, we obtain

$$\frac{\partial \tilde{F}_{;k}}{\partial y^l} y^k = \mu \frac{\partial^2[\tilde{F}^2]}{\partial y^k \partial y^l} y^k = \mu \frac{\partial[\tilde{F}^2]}{\partial y^l} = \tilde{F}_{;l}. \tag{30}$$

By Lemma 2.1, we conclude that $\tilde{F}$ is pointwise projective to F. Contracting (26) with y^k yields

$$\tilde{F}_{;k} y^k = 2\mu \tilde{F}^2. \tag{31}$$

Thus the function P in (19) simplifies to

$$P = \frac{\tilde{F}_{;k} y^k}{2\tilde{F}} = \mu \tilde{F}. \tag{32}$$

Using (23), (31) and (32), we obtain

$$\begin{aligned} \Xi(y) &= \mu^2 \tilde{F}^2 - \mu \tilde{F}_{;k} y^k \\ &= \mu^2 \tilde{F}^2 - 2\mu^2 \tilde{F}^2 = -\mu^2 \tilde{F}^2. \end{aligned}$$

Plugging (32) into (24) yields

$$\tau_y(u) = \mu^2 \tilde{g}_y(y, u).$$

This proves the proposition. Q.E.D.

The Funk metric $F_\pm$ on a strongly convex domain $\Omega \subset \mathrm{R}^n$ satisfy

$$\frac{\partial F_\pm}{\partial x^k} = \pm \frac{1}{2} \frac{\partial[F_\pm^2]}{\partial y^k}. \tag{33}$$

Thus $F_\pm$ are of constant curvature $-1/4$ by Proposition 2.2. Since F_- and F_+ are projective metrics, so is $F_H = \frac{1}{2}(F_- + F_+)$. It follows from (33) that the projective factor P of F_H in (19) is given by

$$P := \frac{[F_H]_{;k} y^k}{2F_H} = \frac{1}{2}(F_+ - F_-),$$

and

$$\Xi := P^2 - P_{;k} y^k = -[F_H]^2.$$

Therefore the Hilbert metric F_H has constant curvature -1. This proof is given by Okada [Ok].

3 Projectively Related Einstein Metrics

Assume that F and $\tilde{F}$ are pointwise projectively related Einstein metrics with

$$\mathbf{Ric}(y) = (n-1)\lambda F^2(y), \qquad \widetilde{\mathbf{Ric}}(y) = (n-1)\tilde{\lambda}\tilde{F}^2(y).$$

Then (17) and (25) hold. It is much easier to work on (25) than (17). Let us write (25) as follows.

$$\tilde{\lambda}\tilde{F}^2 = \lambda F^2 + \frac{3}{4}\Big(\frac{\tilde{F}_{;k}y^k}{\tilde{F}}\Big)^2 - \frac{\tilde{F}_{;k;l}y^ky^l}{2\tilde{F}}. \tag{34}$$

Let $c(t)$ be an arbitrary unit speed geodesic in (M, F) and

$$\tilde{F}(t) := \tilde{F}(\dot{c}(t)).$$

Observe that

$$\tilde{F}'(t) = \tilde{F}_{;k}(\dot{c}(t))\dot{x}^k(t), \qquad \tilde{F}''(t) = \tilde{F}_{;k;l}(\dot{c}(t))\dot{x}^k(t)\dot{x}^l(t).$$

Let

$$f(t) := \frac{1}{\sqrt{\tilde{F}(t)}}.$$

(34) simplifies to

$$f''(t) + \lambda f(t) = \frac{\tilde{\lambda}}{f^3(t)}. \tag{35}$$

The equation (35) is solvable.

For simplicity, let

$$C := \frac{1}{2}\Big(\lambda a^2 + \tilde{\lambda}/a^2 + b^2\Big).$$

The solution of (35) with

$$f(0) = a > 0, \qquad f'(0) = b \neq 0$$

is determined by

$$\int_a^{f(t)} \frac{s}{\sqrt{-\lambda s^4 + 2Cs^2 - \tilde{\lambda}}} ds = \pm t, \tag{36}$$

where the sign $\pm$ in (36) is same as that of $f'(0) = b$. The solution with

$$f(0) = a > 0, \qquad f'(0) = 0$$

can be obtained by letting $b \to 0$.

Note that

$$-\lambda\Big(a^2 - C/\lambda\Big)^2 + C^2/\lambda - \tilde{\lambda} = (ab)^2 > 0, \quad \text{if } \lambda \neq 0 \tag{37}$$

and

$$-\lambda a^4 + 2Ca^2 - \tilde{\lambda} = (ab)^2 > 0. \tag{38}$$

Thus the integrand in (36) is defined for s close to a and the maximal solution $f(t) > 0$ exists on an interval I containing $s = 0$.

4 $\lambda = 1$

In this section, we study the equation (36) when $\lambda = 1$. In this case

$$C = \frac{1}{2}\Big(a^2 + \tilde{\lambda}/a^2 + b^2\Big), \qquad C^2 - \tilde{\lambda} = \Big(a^2 - C\Big)^2 + (ab)^2.$$

From (36), we obtain

$$f(t) = \sqrt{(a^2 - C)\cos(2t) + ab\sin(2t) + C}. \tag{39}$$

We use (37) to rewrite (39) in the following form

$$f(t) = \sqrt{\sqrt{C^2 - \tilde{\lambda}}\,\sin\Big[\sin^{-1}\Big(\frac{a^2 - C}{\sqrt{C^2 - \tilde{\lambda}}}\Big) \pm 2t\Big] + C}, \tag{40}$$

where the sign $\pm$ in (40) is same as that of $f'(0) = b$ when $b \neq 0$. Otherwise, the sign can be chosen arbitrarily.

Case 1: $\tilde{\lambda} = 1$. In this case,

$$C = \frac{1}{2}(a^2 + 1/a^2 + b^2) > 1, \quad C^2 - 1 = (a^2 - C)^2 + (ab)^2,$$

$$\frac{|C|}{\sqrt{C^2 - 1}} > 1.$$

Then

$$f(t) = \sqrt{\sqrt{C^2 - 1}\sin\Big[\sin^{-1}\Big(\frac{a^2 - C}{\sqrt{C^2 - 1}}\Big) \pm 2t\Big] + C}.$$

Thus $f(t)$ is defined on $I = (-\infty, \infty)$ and for any r,

$$\int_r^{r+\pi} \frac{1}{f(t)^2}dt = \pi.$$

Case 2: $\tilde{\lambda} = 0$. In this case,

$$C = \frac{1}{2}(a^2 + b^2) > 0, \qquad C^2 = (a^2 - C)^2 + (ab)^2.$$

Then

$$f(t) = \sqrt{C}\sqrt{\sin\Big[\sin^{-1}\Big(\frac{a^2 - C}{C}\Big) \pm 2t\Big] + 1}.$$

Thus $f(t)$ is defined on a bounded interval $I = (-\delta, \tau)$ and

$$\int_{-\delta}^{0} \frac{1}{f(t)^2}dt = \infty = \int_0^{\tau} \frac{1}{f(t)^2}dt.$$

Case 3: $\tilde{\lambda} = -1$. In this case,

$$C = \frac{1}{2}(a^2 - 1/a^2 + b^2), \qquad C^2 + 1 = (a^2 - C)^2 + (ab)^2,$$

$$\frac{|C|}{\sqrt{C^2+1}} < 1.$$

Then

$$f(t) = \sqrt{\sqrt{C^2+1}\sin\Big[\sin^{-1}\Big(\frac{a^2 - C}{\sqrt{C^2+1}}\Big) \pm 2t\Big] + C}.$$

Thus $f(t)$ is defined on a bounded interval $I = (-\delta, \tau)$ and

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty = \int_0^{\tau} \frac{1}{f(t)^2} dt.$$

From the above arguments, we obtain the following

Proposition 4.1 *Let F and $\tilde{F}$ be Einstein metrics on an n-manifold M with*

$$\mathbf{Ric} = n - 1, \qquad \widetilde{\mathbf{Ric}} = (n-1)\tilde{\lambda}.$$

Then for any unit speed geodesic $c(t)$ of F,

$$\tilde{F}(\dot{c}(t)) = \frac{2}{\Big(a^2 - \tilde{\lambda}/a^2 - b^2\Big)\cos(2t) + 2ab\sin(2t) + \Big(a^2 + \tilde{\lambda}/a^2 + b^2\Big)}. \tag{41}$$

(i) If $\tilde{\lambda} = 1$, then along any unit speed geodesic c of F,

$$\tilde{F}(\dot{c}(t)) = \frac{1}{\sqrt{C^2 - 1}\sin(\theta \pm 2t) + C},$$

where $C > 1$ and $\theta \in [-\pi/2, \pi/2]$. Thus for any unit speed geodesic c of F with F-length $L_F(c) = \pi$, the $\tilde{F}$-length $L_{\tilde{F}}(c) = \pi$.

(ii) If $\tilde{\lambda} = 0$, then along any unit speed geodesic c of F,

$$\tilde{F}(\dot{c}(t)) = \frac{1}{C\sin(\theta \pm 2t) + C},$$

where $C > 0$ and $\theta \in [-\pi/2, \pi/2]$. Thus every geodesic of F has finite length.

(iii) If $\tilde{\lambda} = -1$, the along any unit speed geodesic c of F,

$$\tilde{F}(\dot{c}(t)) = \frac{1}{\sqrt{C^2 + 1}\sin(\theta \pm 2t) + C},$$

where C is a constant and $\theta \in [-\pi/2, \pi/2]$. Thus every geodesic of F has finite length.

Consider the following spherical metric on R^n:

$$F_S(y) := \frac{\sqrt{|y|^2 + \left(|x|^2|y|^2 - \langle x, y\rangle^2\right)}}{1+|x|^2}, \tag{42}$$

F_S is of constant curvature 1 and pointwise projective to the standard complete Euclidean metric F_E on R^n. More over, all geodesics (straight lines) have F_S-length $L_{F_S}(c) = \pi$.

5 $\lambda = 0$

In this section, we shall study the equation (36) when $\lambda = 0$. From (36) we obtain

$$f(t) = \sqrt{\left(a+bt\right)^2 + \tilde{\lambda}\left(\frac{t}{a}\right)^2}. \tag{43}$$

Case 1: $\tilde{\lambda} = 1$. In this case, Then

$$f(t) = \sqrt{\left(a+bt\right)^2 + \left(\frac{t}{a}\right)^2}.$$

Thus $f(t)$ is defined on $I = (-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} \frac{1}{f(t)^2} dt = \pi.$$

Case 2: $\tilde{\lambda} = 0$. In this case,

$$f(t) = a + bt.$$

(i) If $b = 0$, then

$$f(t) = a.$$

Thus $f(t)$ is defined on $I = (-\infty, \infty)$.

(ii) If $b \neq 0$, then

$$f(t) = a + bt.$$

In this case when $b > 0$, $I = (-\delta, \infty)$ and

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $b < 0$ is similar, so is omitted.

Case 3: $\tilde{\lambda} = -1$. In this case,

$$f(t) = \sqrt{\left(a+bt\right)^2 - \left(\frac{t}{a}\right)^2}.$$

(i) If $ab=1$, then

$$f(t)=\sqrt{a^2+2t}.$$

In this case, $I=(-\delta,\infty)$ and

$$\int_{-\delta}^{0}\frac{1}{f(t)^2}dt=\infty=\int_0^{\infty}\frac{1}{f(t)^2}dt.$$

The case when $ab=-1$ is similar.

(ii) If $-1<ab<1$, then $f(t)$ is defined on a bounded interval $(-\delta,\tau)$ and Clearly,

$$\int_{-\delta}^{0}\frac{1}{f(t)^2}dt=\infty \quad \text{and} \quad \int_0^{\tau}\frac{1}{f(t)^2}dt=\infty.$$

(iii) If $ab>1$, then $f(t)$ is defined on $(-\delta,\infty)$ and

$$\int_{-\delta}^{0}\frac{1}{f(t)^2}dt=\infty \quad \text{and} \quad \int_0^{\infty}\frac{1}{f(t)^2}dt<\infty.$$

The case when $ab<-1$ is similar, so is omitted.

Proposition 5.1 *Let F and $\tilde{F}$ be Einstein metrics on an n-manifold M with*

$$\mathbf{Ric}=0, \qquad \widetilde{\mathbf{Ric}}=(n-1)\tilde{\lambda}.$$

Assume that F and $\tilde{F}$ are pointwise projectively related on M. Then for any unit speed geodesic $c(t)$ of F,

$$\tilde{F}(\dot{c}(t))=\frac{1}{\Big(a+bt\Big)^2+\tilde{\lambda}\Big(\frac{t}{a}\Big)^2}. \tag{44}$$

(i) If $\tilde{\lambda}=1$, then along any geodesic $c(t)$ of F,

$$\tilde{F}(\dot{c}(t))=\frac{1}{\Big(a+bt\Big)^2+\Big(\frac{t}{a}\Big)^2}. \tag{45}$$

Thus for any geodesic c of F, the $\tilde{F}$-length $L_{\tilde{F}}(c)\le\pi$. Equality holds when F is complete.

(ii) If $\tilde{\lambda}=0$, then along any unit speed geodesic $c(t)$ of F,

$$\tilde{F}(\dot{c}(t))=\frac{1}{(a+bt)^2}.$$

(iia) If a unit speed geodesic c of F is defined on $(-\infty,\infty)$, then $\tilde{F}(\dot{c}(t))=1/a^2$.

(iib) If a unit speed geodesic c of F is defined on $[0,\infty)$, then it has finite $\tilde{F}$-length unless $\tilde{F}(\dot{c}(t))=1/a^2$.

(iib) If a unit speed geodesic c of F is defined on $(-\infty, 0]$, then it has finite $\tilde{F}$-length unless $\tilde{F}(\dot{c}(t)) = 1/a^2$.
Therefore, F is complete if and only if $\tilde{F}$ is complete. In this case, along any geodesic c

$$\frac{F(\dot{c}(t))}{\tilde{F}(\dot{c}(t))} = \textit{constant}.$$

(iii) If $\tilde{\lambda} = -1$, then along any unit speed geodesic c of F,

$$\tilde{F}(\dot{c}(t)) = \frac{1}{\Big(a + bt\Big)^2 - \Big(\frac{t}{a}\Big)^2}. \tag{46}$$

Thus no geodesic of F is defined on $(-\infty, \infty)$.

(iiia) If a unit speed geodesic c of F is defined on $[0, \infty)$, then it has finite $\tilde{F}$-length unless

$$\tilde{F}(\dot{c}(t)) = \frac{1}{a^2 + 2t}. \tag{47}$$

(iiib) If a unit speed geodesic c of F is defined on $(-\infty, 0]$, then it has finite $\tilde{F}$-length unless

$$\tilde{F}(\dot{c}(t)) = \frac{1}{a^2 - 2t}. \tag{48}$$

For the spherical metric F_S in (42), the geodesics of F_S are straight lines in R^n. Thus it is pointwise projective to the standard Euclidean metric F_E on R^n. It is easy to verify that all lines in R^n have length of π with respect to F_S. This is also guaranteed by Proposition 5.1 (i).

6 $\lambda = -1$

In this section, we shall study the equation (36) when $\lambda = -1$. In this case,

$$C = \frac{1}{2}\Big(- a^2 + \tilde{\lambda}/a^2 + b^2\Big), \qquad \Big(a^2 + C\Big)^2 = C^2 + \tilde{\lambda} + (ab)^2.$$

From (36), we obtain

$$f(t) = \sqrt{(a^2 + C)\cosh(2t) + ab\sinh(2t) - C}. \tag{49}$$

We use (37) to rewrite (49) as follows

$$f(t) = \begin{cases} \sqrt{\sqrt{C^2 + \tilde{\lambda}}\ \cosh\Big[\cosh^{-1}\Big(\frac{a^2+C}{\sqrt{C^2+\tilde{\lambda}}}\Big) \pm 2t\Big] - C} & \text{if } C^2 + \tilde{\lambda} > 0 \\ \sqrt{e^{\pm 2t}\Big(a^2 + C\Big) - C} & \text{if } C^2 + \tilde{\lambda} = 0 \\ \sqrt{\sqrt{-C^2 - \tilde{\lambda}}\ \sinh\Big[\sinh^{-1}\Big(\frac{a^2+C}{\sqrt{-C^2-\tilde{\lambda}}}\Big) \pm 2t\Big] - C} & \text{if } C^2 + \tilde{\lambda} < 0 \end{cases} \tag{50}$$

The sign $\pm$ in (50) is same as that of $f'(0) = b \neq 0$.

We divide this case into several cases.

Case 1: $\tilde{\lambda} = 1$. In this case,

$$C = \frac{1}{2}\Big(- a^2 + 1/a^2 + b^2\Big), \qquad (a^2 + C)^2 = C^2 + 1 + (ab)^2,$$

$$\frac{|C|}{\sqrt{C^2+1}} < 1.$$

Then

$$f(t) = \sqrt{\sqrt{C^2+1}\,\cosh\Big[\cosh^{-1}\Big(\frac{a^2+C}{\sqrt{C^2+1}}\Big) \pm 2t\Big] - C}.$$

Thus $f(t)$ is defined on $I = (-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

Case 2: $\tilde{\lambda} = 0$. In this case,

$$C = \frac{1}{2}(-a^2 + b^2), \qquad (a^2 + C)^2 = C^2 + (ab)^2.$$

Then

$$f(t) = \begin{cases} \sqrt{2|C|}\cosh\Big[\frac{1}{2}\cosh^{-1}\Big(\frac{a^2}{|C|} - 1\Big) \pm t\Big] & \text{if } C < 0, \\ \sqrt{2C}\sinh\Big[\frac{1}{2}\cosh^{-1}\Big(\frac{a^2}{C} + 1\Big) \pm t\Big] & \text{if } C > 0, \\ ae^{\pm t} & \text{if } C = 0\ . \end{cases}$$

(i) If $C < 0$, then $f(t)$ is defined on $I = (-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

(ii) If $C > 0$, then $f(t)$ is defined on either $I = (-\delta, \infty)$ or $I = (-\infty, \tau)$. Assume that $I = (-\delta, \infty)$. Then

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $I = (-\infty, \tau)$ is similar, so is omitted.

(iii) If $C = 0$, then $b \neq 0$ and $f(t)$ is defined on $I = (-\infty, \infty)$. Assume that $b > 0$. Then

$$\int_{-\infty}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $b < 0$ is similar, so is omitted.

Case 3: $\tilde{\lambda} = -1$. In this case,

$$C = \frac{1}{2}\Big(- a^2 - 1/a^2 + b^2\Big), \qquad (a^2 + C)^2 = C^2 - 1 + (ab)^2.$$

(i) $C^2 > 1$. In this case,

$$\frac{|C|}{\sqrt{C^2 - 1}} > 1.$$

Then

$$f(t) = \sqrt{\sqrt{C^2-1}\cosh\Big[\cosh^{-1}\Big(\frac{a^2+C}{\sqrt{C^2-1}}\Big) \pm 2t\Big] - C}.$$

(ia) If $C > 1$, then $f(t)$ is defined on $I = (-\delta, \infty)$ and

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

(ib)If $C < -1$, then $f(t)$ is defined on $I = (-\infty, \infty)$ and

$$\int_{-\infty}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

(ii) $C^2 < 1$. Then

$$f(t) = \sqrt{\sqrt{1-C^2}\sinh\Big[\sinh^{-1}\Big(\frac{a^2+C}{1-C^2}\Big) \pm 2t\Big] - C}.$$

In this case, $f(t)$ is defined on either $I = (-\delta, \infty)$ or $I = (-\infty, \tau)$. Assume that $I = (-\delta, \infty)$. Then

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $I = (-\infty, \tau)$ is similar, so is omitted.

(iii) $C^2 = 1$.

(iiia) If $C = 1$, then

$$f(t) = \sqrt{e^{\pm 2t}(a^2 + 1) - 1}.$$

Thus $f(t)$ is defined on either $I = (-\delta, \infty)$ or $I = (-\infty, \tau)$. Assume that $I = (-\delta, \infty)$. Then

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $I = (-\infty, \tau)$ is similar, so is omitted.

(iiib) If $C = -1$, then

$$f(t) = \sqrt{e^{\pm 2t}(a^2 - 1) + 1}.$$

If $a > 1$, then $b \neq 0$ and $f(t)$ is defined on $I = (-\infty, \infty)$. Assume that $b > 0$, then

$$\int_{-\infty}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $b < 0$ is similar, so is omitted.

If $0 < a < 1$, then $f(t)$ is defined on either $I = (-\delta, \infty)$ or $I = (-\infty, \tau)$. Assume that $I = (-\delta, \infty)$. Then

$$\int_{-\delta}^{0} \frac{1}{f(t)^2} dt = \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{f(t)^2} dt < \infty.$$

The case when $b > 0$ is similar, so is omitted.

If $a = 1$, then $b = 0$. Then

$$f(t) = 1.$$

In this case, $f(t)$ is defined on $I = (-\infty, \infty)$.

Proposition 6.1 *Let F and $\tilde{F}$ be Einstein metrics on an n-manifold M with*

$$\mathbf{Ric} = -(n-1), \qquad \widetilde{\mathbf{Ric}} = (n-1)\tilde{\lambda}.$$

Assume that F and $\tilde{F}$ are pointwise projectively related. Then for any geodesic of $c(t)$ of F,

$$\tilde{F}(\dot{c}(t)) = \frac{1}{\Big(a^2 + \tilde{\lambda}/a^2 + b^2\Big)\cosh(2t) + 2ab\sinh(2t) - \Big(-a^2 + \tilde{\lambda}/a^2 + b^2\Big)}. \tag{51}$$

(i) $\tilde{\lambda} = 1$. In this case, any geodesic of $\tilde{F}$ has finite length. Hence $\tilde{F}$ is neither positively complete, nor negatively complete.

(ii) $\tilde{\lambda} = 0$. In this case, no geodesic of $\tilde{F}$ is defined on $(-\infty, \infty)$.

(iia) If a unit speed geodesic c of F is defined on $[0, \infty)$, then it has finite $\tilde{F}$-length unless

$$\tilde{F}(\dot{c}) = \Big(\frac{e^t}{a}\Big)^2. \tag{52}$$

(iib) If a unit speed geodesic c of F is defined on $(-\infty, 0]$, then it has finite $\tilde{F}$-length unless

$$\tilde{F}(\dot{c}(t)) = \Big(\frac{e^{-t}}{a}\Big)^2. \tag{53}$$

(iii) $\tilde{\lambda} = -1$. In this case, if both F and $\tilde{F}$ are complete, then

$$F = \tilde{F}.$$

(iiia) If a unit speed geodesic c of F is defined on $[0,\infty)$, then it has finite $\tilde{F}$-length unless

$$\tilde{F}(\dot{c}(t)) = \frac{1}{e^{-2t}(a^2-1)+1}, \qquad (a \geq 1). \tag{54}$$

(iiib) If a unit speed geodesic c of F is defined on $(-\infty,0]$, then it has finite $\tilde{F}$-length unless

$$\tilde{F}(\dot{c}(t)) = \frac{1}{e^{2t}(a^2-1)+1}, \qquad (a \geq 1). \tag{55}$$

7 Examples

Below are some interesting examples. All the metrics are projective Finsler metrics of constant curvature on a strongly convex domain in the Euclidean space.

Example 7.1 The standard metric on the upper/lower semi-sphere $\mathrm{S}^n_\pm$ can be pulled back to the spherical metric F_S on R^n by a diffeomorphism $\varphi_\pm : \mathrm{R}^n \to \mathrm{S}^n_\pm$,

$$\varphi_\pm(x) := \Big(\frac{x}{\sqrt{1+|x|^2}}, \frac{\pm 1}{\sqrt{1+|x|^2}} \Big). \tag{56}$$

The formula of F_S is given in (42). F_S has positive constant curvature $= 1$ and it is pointwise projective to the standard flat metric $F_E(y) = |y|$ on R^n. Take an arbitrary geodesic $c(t) = x + ty$ in (R^n, F). Then

$$\begin{aligned} F_S(\dot{c}(t)) &= \frac{\sqrt{|y|^2 + \Big(|x|^2|y|^2 - \langle x,y\rangle^2 \Big)}}{1+|x|^2+2\langle x,y\rangle t + |y|^2t^2} && (57) \\ &= \frac{1}{\Big(a+bt \Big)^2 + \Big(\frac{t}{a} \Big)^2}, && (58) \end{aligned}$$

where

$$\begin{aligned} a &= \frac{\sqrt{1+|x|^2}}{\Big[|y|^2 + \Big(|x|^2|y|^2 - \langle x,y\rangle^2 \Big]^{1/4}} \\ b &= \frac{\langle x,y\rangle}{\sqrt{1+|x|^2}\Big[|y|^2 + \Big(|x|^2|y|^2 - \langle x,y\rangle^2 \Big]^{1/4}} \end{aligned}$$

Thus $F_S(\dot{c}(t))$ is in the form (45).

Example 7.2 Deforming the spherical metric F_S yields some interesting Finsler metrics. Let

$$
\begin{aligned}
A_\varepsilon(y): &= |x|^2|y|^2-\langle x,y\rangle^2+\varepsilon\,|y|^2+\frac{2(1-\varepsilon^2)\langle x,y\rangle^2}{|x|^4+2\varepsilon|x|^2+1},\\
B_\varepsilon(y): &= \Big(|x|^2|y|^2-\langle x,y\rangle^2\Big)^2+2\varepsilon\,\Big(|x|^2|y|^2-\langle x,y\rangle^2\Big)|y|^2+|y|^4.
\end{aligned}
$$

For $0<\varepsilon\le 1$, define

$$
F_\varepsilon(y):=\sqrt{\frac{A_\varepsilon(y)+\sqrt{B_\varepsilon(y)}}{2\Big(|x|^4+2\varepsilon|x|^2+1\Big)}}+\frac{\sqrt{1-\varepsilon^2}\langle x,y\rangle}{|x|^4+2\varepsilon\,|x|^2+1},\quad y\in T_x\mathrm{R}^n. \tag{59}
$$

F_ε is a family of Finsler metrics on R^n. Note that $F_1=F_S$ is just the spherical metric in (42). Using (56), one can pull F_ε onto S^n. The pull-back metrics on S^n are the natural generalization of Bryant metrics on S^2 [Br1][Br2].

Assume that F_ε are of constant curvature 1 and pointwise projective to the Euclidean metric F_E on R^n. Then for any $c(t)=x+ty$, there are constants $a>0$ and $-\infty<b<\infty$ such that

$$
F_\varepsilon(\dot c(t))=\frac{1}{\Big(a+bt\Big)^2+\Big(\frac{t}{a}\Big)^2}.
$$

The constants a and b must be given by

$$
a=\frac{1}{\sqrt{F_\varepsilon(y)}},\qquad b=-\frac{y^i}{\sqrt{F_\varepsilon(y)}}\frac{\partial}{\partial x^i}\Big[\ln\sqrt{F_\varepsilon(y)}\Big].
$$

The proof maybe need a faster computer.

Example 7.3 The Klein metric F_K in (2) is Riemannian. It is complete with constant curvature -1 and pointwise projective to the standard (incomplete) Euclidean metric $F_E(y)=|y|$ on B^n. Take an arbitrary geodesic $c(t)=x+ty$ in (B^n,F). Then

$$
\begin{aligned}
F_K(\dot c(t)) &= \frac{\sqrt{|y|^2-\Big(|x|^2|y|^2-\langle x,y\rangle^2\Big)}}{1-|x|^2-2\langle x,y\rangle t-|y|^2t^2}\\
&= \frac{1}{\Big(a+bt\Big)^2-\Big(\frac{t}{a}\Big)^2},
\end{aligned}\tag{60}
$$

where

$$
\begin{aligned}
a &= \frac{\sqrt{1-|x|^2}}{\Big[|y|^2-\Big(|x|^2|y|^2-\langle x,y\rangle^2\Big)\Big]^{1/4}}\\
b &= -\frac{\langle x,y\rangle}{\sqrt{1-|x|^2}\Big[|y|^2-\Big(|x|^2|y|^2-\langle x,y\rangle^2\Big)\Big]^{1/4}}.
\end{aligned}
$$

Thus $F_K(\dot{c}(t))$ are in the form (46).

Example 7.4 (Funk metrics) Let Ω be a strongly convex bounded domain in R^n. For $0 \neq y \in T_x\Omega \approx \mathrm{R}^n$, let $F_-(y) > 0$ and $F_+(y) > 0$ be given by

$$z_- = x - \frac{y}{F_-(y)}, \qquad z_+ = x + \frac{y}{F_+(y)}, \tag{61}$$

where z_-, z_+ are the intersection points of the line $\ell(t) := x + ty$ with $\partial\Omega$ such that $z_+ - z_-$ is in the same direction as y. $F_\pm$ are called the pair of *Funk metrics* on Ω. Note that $F_+(-y) = F_-(y)$. More over, they satisfy

$$\frac{\partial F_\pm}{\partial x^k} = \pm \frac{1}{2} \frac{\partial [F_\pm]}{\partial y^k}.$$

According to Proposition 2.1, $F_\pm$ are of constant curvature $-1/4$ and pointwise projective to the standard Euclidean metric F_E on Ω. This simple proof is due to T. Okada [Ok].

Fix $y \in T_x\Omega$ and t such that $c(t) = x + ty \in \Omega$. From the definition of F_- and F_+, we have

$$z_- = x - \frac{y}{F_-(y)} = x + ty - \frac{y}{F_-(\dot{c}(t))}.$$

$$z_+ = x + \frac{y}{F_+(y)} = x + ty + \frac{y}{F_+(\dot{c}(t))}.$$

Then we obtain

$$F_-(\dot{c}(t)) = \frac{F_-(y)}{1 + F_-(y)t}, \tag{62}$$

$$F_+(\dot{c}(t)) = \frac{F_+(y)}{1 - F_+(y)t}. \tag{63}$$

Thus

$$\frac{1}{2} F_\pm(\dot{c}(t)) = \frac{1}{a^2 \mp 2t},$$

where $a^2 = 2/F_\pm(y)$. Thus $\frac{1}{2}F_\pm(\dot{c}(t))$ are in the form (46) with $ab = \mp 1$. This is also guaranteed by Proposition 5.1, because that $\frac{1}{2}F_\pm$ has constant curvature -1.

Example 7.5 Let Ω be a strongly convex bounded domain in R^n. Let $F_\pm$ denote the Funk metrics on Ω defined in (61). Define

$$F_H(y) := \frac{1}{2}\Big(F_-(y) + F_+(y)\Big). \tag{64}$$

F_H is called the *Hilbert metric* on Ω. The Hilbert is of constant curvature -1 and pointwise projective to the Euclidean metric F_E on Ω. See [Bu] [BuKe] [Fk] [Ok].

It follows from Proposition 5.1 that along any geodesic $c(t) = x + ty$ of F_E, $F_H(\dot{c}(t))$ should satisfy (46). Let us verify this necessary condition directly. From (62) and (63), we obtain

$$\begin{aligned} F_H(\dot{c}(t)) &= \frac{F_-(y) + F_+(y)}{2\Big(1 + F_-(y)t\Big)\Big(1 - F_+(y)t\Big)} \\ &= \frac{1}{\Big(a + bt\Big)^2 - \Big(\frac{t}{a}\Big)^2}, \end{aligned}$$

where

$$a = \frac{\sqrt{2}}{\sqrt{F_-(y) + F_+(y)}}, \qquad b = \frac{F_-(y) - F_+(y)}{\sqrt{2}\sqrt{F_-(y) + F_+(y)}}.$$

Yes, $F_H(\dot{c}(t))$ satisfies (46).

Let $F = F_H$ and $\tilde{F} = F_\pm$. Then along any unit speed geodesic $c(t)$ of F, $\tilde{F}_+(\dot{c}(t))$ satisfies (54) and $\tilde{F}_-(\dot{c}(t))$ satisfies (55).

Example 7.6 Let Ω denote the domain above the graph $x^n = \sum_{a=1}^{n-1}(x^a)^2$ in R^n. Define $\tilde{F} : T\Omega \to [0, \infty)$ by

$$\tilde{F}(y) := \frac{\sqrt{\Big(y^n - 2\sum_{a=1}^{n-1} x^a y^a\Big)^2 + 4\Big(x^n - \sum_{a=1}^{n-1}(x^a)^2\Big)\sum_{a=1}^{n-1}(y^a)^2}}{2\Big(x^n - \sum_{a=1}^{n}(x^a)^2\Big)}. \tag{65}$$

This Riemann metric has constant curvature -1. Take an arbitrary geodesic $c(t) = x + ty$ in (Ω, F), $\tilde{F}(\dot{c}(t))$ must be in the form (46). Note that every geodesic $c(t) = x + ty$ must intersect $\partial\Omega$ on both sides unless $y = (0, \cdots, 0, y^n)$ with $y^n > 0$. When $y = (0, \cdots, 0, y^n)$ with $y^n > 0$, the geodesic $c(t) = x + ty$ of F is defined on $(-\delta, \infty)$ and the $\tilde{F}$-length of c over $[0, \infty)$ is finite. Moreover,

$$\tilde{F}(\dot{c}(t)) = \frac{1}{a^2 + 2t},$$

where

$$a^2 = 2\frac{x^n - \sum_{a=1}^{n-1}(x^a)^2}{y^n}.$$

The last statement is also implied by Proposition 5.1 (iiia).

Finally we ask the following

Open Problem: Are there non-trivial positively/negatively complete Ricci-flat metrics on an open subset $\Omega \subset \mathrm{R}^n$? If any, they must be R-flat (i.e., $\mathbf{R} = 0$).

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